A hyponormal weighted shift on a directed tree whose square has trivial domain

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Abstract. It is proved that, up to isomorphism, there are only two directed trees that admit a hyponormal weighted shift with nonzero weights whose square has trivial domain. These are precisely those enumerable directed trees, one with root, the other without, whose every vertex has enumerable set of successors.

1. Introduction

In a recent paper [**4**] a question of subnormality of unbounded weighted shifts on directed trees has been investigated. A criterion for subnormality of such operators whose C^∞-vectors are dense in the underlying Hilbert space has been established (cf. [**4**, Theorem 5.2.1]). It has been written in terms of consistent systems of Borel probability measures. The assumption that the operator in question has a dense set of C^∞-vectors diminishes the class of weighted shifts on directed trees to which this criterion can be applied (note that the set of all C^∞-vectors of a classical, unilateral or bilateral, weighted shift is always dense in the underlying Hilbert space). Unfortunately, there is no general *criterion* for subnormality of densely defined operators that have small set of C^∞-vectors. The known characterizations of subnormality of unbounded Hilbert space operators require the existence of additional objects (like semispectral measures, elementary spectral measures or sequences of unbounded operators) that have to satisfy appropriate, more or less complicated, conditions (cf. [**3, 7, 20, 21**]). Among subnormal operators having small set of C^∞-vectors, the symmetric ones (which are always subnormal, see [**1**, Theorem 1 in Appendix I.2]) play an essential role. According to [**13**] (see also [**5**]) there are closed symmetric operators whose squares have trivial domain. Unfortunately, symmetric weighted shifts on directed trees are automatically bounded; the same is true for formally normal weighted shifts on directed trees (cf. [**9**, Proposition 3.1]).

The above discussion leads to the following problem.

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Question. Does there exist a subnormal weighted shift on a directed tree with nonzero weights whose square has trivial domain?

At present, this question is unanswered (the reason for this is explained partially in the previous paragraph). However, as is shown in Theorem 4.2, there are injective hyponormal weighted shifts on directed trees with nonzero weights whose squares have trivial domain. What is more, it is proved in Theorem 3.1 that the only directed trees admitting densely defined weighted shifts with nonzero weights whose squares have trivial domain are those enumerable directed trees whose every vertex has enumerable set of successors (children).

2. Preliminaries

In what follows, $\mathbb{C}$ stands for the set of all complex numbers. Let A be an operator in a complex Hilbert space $\mathcal{H}$ (all operators considered in this paper are linear). Denote by $\mathcal{D}(A)$ and A^* the domain and the adjoint of A (in case it exists). A closed densely defined operator N in $\mathcal{H}$ is called *normal* if $N^*N = NN^*$. A densely defined operator S in $\mathcal{H}$ is said to be *subnormal* if there exists a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Sh = Nh$ for all $h \in \mathcal{D}(S)$. Finally, a densely defined operator S in $\mathcal{H}$ is called *hyponormal* if $\mathcal{D}(S) \subseteq \mathcal{D}(S^*)$ and $\|S^*f\| \leqslant \|Sf\|$ for all $f \in \mathcal{D}(S)$. It is well-known that subnormal operators are hyponormal (but not conversely) and that hyponormal operators are closable and their closures are hyponormal (subnormal operators have an analogous property). We refer the reader to [**2, 22**] for basic facts on unbounded operators, [**6, 16, 17, 18, 19**] for the foundations of the theory of (bounded and unbounded) subnormal operators and [**14, 10, 11, 12, 15**] for elements of the theory of unbounded hyponormal operators.

Let $\mathscr{T} = (V, E)$ be a directed tree (V and E always stand for the sets of vertices and edges of $\mathscr{T}$, respectively). If $\mathscr{T}$ has a root, which will always be denoted by root, then we write $V^\circ := V \setminus \{\mathsf{root}\}$; otherwise, we put $V^\circ = V$. Set $\mathsf{Chi}(u) = \{v \in V\colon (u, v) \in E\}$ for $u \in V$. If for a given vertex $u \in V$ there exists a unique vertex $v \in V$ such that $(v, u) \in E$, then we denote it by $\mathsf{par}(u)$. The correspondence $u \mapsto \mathsf{par}(u)$ is a partial function from V to V. For an integer $n \geqslant 1$, the n-fold composition of the partial function par with itself will be denoted by par^n. Let par^0 stand for the identity map on V. We call $\mathscr{T}$ *leafless* if $V = \{u \in V\colon \mathsf{Chi}(u) \neq \varnothing\}$. If $W \subseteq V$, we put $\mathsf{Chi}(W) = \bigcup_{v\in W} \mathsf{Chi}(v)$ and $\mathsf{Des}(W) = \bigcup_{n=0}^\infty \mathsf{Chi}^{\langle n\rangle}(W)$, where $\mathsf{Chi}^{\langle 0\rangle}(W) = W$ and $\mathsf{Chi}^{\langle n+1\rangle}(W) = \mathsf{Chi}(\mathsf{Chi}^{\langle n\rangle}(W))$ for all integers $n \geqslant 0$. For $u \in V$, we set $\mathsf{Chi}^{\langle n\rangle}(u) = \mathsf{Chi}^{\langle n\rangle}(\{u\})$ and $\mathsf{Des}(u) = \mathsf{Des}(\{u\})$. Combining equalities (2.1.3), (6.1.3) and (2.1.10) of [**8**] with [**8**, Corollary 2.1.5], we obtain

$$V^\circ = \bigsqcup_{u\in V} \mathsf{Chi}(u), \tag{2.1}$$

$$\mathsf{Chi}^{\langle n+1\rangle}(u) = \bigsqcup_{v\in \mathsf{Chi}^{\langle n\rangle}(u)} \mathsf{Chi}(v), \quad u \in V,\ n = 0, 1, 2, \ldots, \tag{2.2}$$

$$\mathsf{Des}(u) = \bigsqcup_{n=0}^{\infty} \mathsf{Chi}^{\langle n\rangle}(u), \quad u \in V, \tag{2.3}$$

$$\mathsf{Des}(u_1) \cap \mathsf{Des}(u_2) = \varnothing, \quad u_1, u_2 \in \mathsf{Chi}(u),\ u_1 \neq u_2,\ u \in V, \tag{2.4}$$

$$V = \mathsf{Des}(\mathsf{root}) \quad \text{provided that } \mathscr{T} \text{ has a root}, \tag{2.5}$$

where the symbol $\bigsqcup$ is reserved to denote pairwise disjoint union of sets.

Let $\ell^2(V)$ be the Hilbert space of all square summable complex functions on V equipped with the standard inner product. For $u \in V$, we define $e_u \in \ell^2(V)$ to be the characteristic function of the one point set $\{u\}$. The family $\{e_u\}_{u\in V}$ is an orthonormal basis of $\ell^2(V)$. Denote by $\mathscr{E}_V$ the linear span of $\{e_u \colon u \in V\}$. Given $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq \mathbb{C}$, we define the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ by

$$\begin{aligned} \mathcal{D}(S_{\boldsymbol{\lambda}}) &= \{f \in \ell^2(V) \colon \varLambda_{\mathscr{T}} f \in \ell^2(V)\}, \\ S_{\boldsymbol{\lambda}} f &= \varLambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}), \end{aligned}$$

where $\varLambda_{\mathscr{T}}$ is the map defined on functions $f \colon V \to \mathbb{C}$ via

$$(\varLambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathsf{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}. \end{cases} \tag{2.6}$$

$S_{\boldsymbol{\lambda}}$ is called a *weighted shift* on the directed tree $\mathscr{T}$ with weights $\{\lambda_v\}_{v\in V^\circ}$. Note that any weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$ is a closed operator (cf. [**8**, Proposition 3.1.2]). Combining Propositions 3.1.3, 3.1.7 and 3.1.10 of [**8**], we get the following fact (hereafter we adopt the convention that $\sum_{v\in\varnothing} x_v = 0$).

Proposition 2.1. *Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$. Then the following assertions hold*:

(i) *e_u is in $\mathcal{D}(S_{\boldsymbol{\lambda}})$ if and only if $\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 < \infty$; if $e_u \in \mathcal{D}(S_{\boldsymbol{\lambda}})$, then*

$$S_{\boldsymbol{\lambda}} e_u = \sum_{v\in\mathsf{Chi}(u)} \lambda_v e_v \quad \text{and} \quad \|S_{\boldsymbol{\lambda}} e_u\|^2 = \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2, \tag{2.7}$$

(ii) *$S_{\boldsymbol{\lambda}}$ is densely defined if and only if $\mathscr{E}_V \subseteq \mathcal{D}(S_{\boldsymbol{\lambda}})$,*

(iii) *$S_{\boldsymbol{\lambda}}$ is injective if and only if $\mathscr{T}$ is leafless and $\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 > 0$ for every $u \in V$,*

(iv) *if $\overline{\mathcal{D}(S_{\boldsymbol{\lambda}})} = \ell^2(V)$ and $\lambda_v \neq 0$ for all $v \in V^\circ$, then V is at most countable.*

3. Directed trees admitting $S_{\boldsymbol{\lambda}}$'s with $\mathcal{D}(S^2_{\boldsymbol{\lambda}}) = \{0\}$

The proof of Theorem 3.1 below contains a method of constructing densely defined weighted shifts $S_{\boldsymbol{\lambda}}$ on directed trees with nonzero weights such that $\mathcal{D}(S^2_{\boldsymbol{\lambda}}) = \{0\}$. By imposing carefully tailored restrictions on weights, we will show in Theorem 4.2 below how to use this method to construct hyponormal weighted shifts on directed trees with the aforesaid properties.

Theorem 3.1. *Let $\mathscr{T}$ be a directed tree. Then the following assertions are equivalent*:

(i) *there exists a family $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$ of nonzero complex numbers such that $\overline{\mathcal{D}(S_{\boldsymbol{\lambda}})} = \ell^2(V)$ and $\mathcal{D}(S^2_{\boldsymbol{\lambda}}) = \{0\}$,*

(ii) $\mathrm{card}(\mathsf{Chi}(u)) = \aleph_0$ *for every $u \in V$.*

Moreover, if $S_{\boldsymbol{\lambda}}$ is as in (i)*, then $S_{\boldsymbol{\lambda}}$ is injective.*

Proof. Fix $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq \mathbb{C}$. We show that

(†) a complex function f on V belongs to $\mathcal{D}(S^2_{\boldsymbol{\lambda}})$ if and only if[1]

$$\sum_{u\in V} \Big(1 + \zeta_u^2 + \sum_{v\in\mathsf{Chi}(u)} \zeta_v^2 |\lambda_v|^2\Big) |f(u)|^2 < \infty, \tag{3.1}$$

[1] with the convention that $0 \cdot \infty = 0$

where $\zeta_u := \sqrt{\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2}$ for $u \in V$.

Indeed, by [**8**, Proposition 3.1.3], a complex function f on V belongs to $\mathcal{D}(S_{\boldsymbol\lambda})$ if and only if $f \in \ell^2(V)$ and $\sum_{u\in V} \zeta_u^2 |f(u)|^2 < \infty$. Hence a complex function f on V belongs to $\mathcal{D}(S_{\boldsymbol\lambda}^2)$ if and only if $\sum_{u\in V}(1+\zeta_u^2)|f(u)|^2 < \infty$ and $\sum_{u\in V} \zeta_u^2\, |(S_{\boldsymbol\lambda} f)(u)|^2 < \infty$. Since the following equalities hold for $f \in \mathcal{D}(S_{\boldsymbol\lambda})$,

$$\begin{aligned}\sum_{u\in V} \zeta_u^2\, |(S_{\boldsymbol\lambda} f)(u)|^2 &\overset{(2.6)}{=} \sum_{u\in V^\circ} \zeta_u^2\, |\lambda_u|^2 |f(\mathsf{par}(u))|^2 \\ &\overset{(2.1)}{=} \sum_{u\in V} \sum_{v\in\mathsf{Chi}(u)} \zeta_v^2 |\lambda_v|^2 |f(\mathsf{par}(v)|^2 \\ &= \sum_{u\in V} \Big(\sum_{v\in\mathsf{Chi}(u)} \zeta_v^2 |\lambda_v|^2 \Big) |f(u)|^2,\end{aligned}$$

we see that a complex function f on V belongs to $\mathcal{D}(S_{\boldsymbol\lambda}^2)$ if and only if (3.1) holds.

(i)$\Rightarrow$(ii) Let $S_{\boldsymbol\lambda}$ be as in (i). By Proposition 2.1(iv), V is countable. Thus each $\mathsf{Chi}(u)$ is countable. Suppose that, contrary to our claim, (ii) does not hold. Then there exists $u_0 \in V$ such that $\mathsf{Chi}(u_0)$ is finite. Since $S_{\boldsymbol\lambda}$ is densely defined, we infer from assertions (i) and (ii) of Proposition 2.1 that $\zeta_v < \infty$ for all $v \in V$. Hence

$$1 + \zeta_{u_0}^2 + \sum_{v\in\mathsf{Chi}(u_0)} \zeta_v^2 |\lambda_v|^2 < \infty.$$

This, combined with (†), implies that $f = e_{u_0} \in \mathcal{D}(S_{\boldsymbol\lambda}^2)$, which contradicts (i).

(ii)$\Rightarrow$(i) First, we show that

(‡) for each $(\vartheta, u) \in (0,\infty) \times V$ there exists $\{\lambda_v\}_{v\in\mathsf{Des}(u)} \subseteq (0,\infty)$ such that

$$\lambda_u^2 = \vartheta, \tag{3.2}$$

$$\Big(\sum_{w\in\mathsf{Chi}(v)} \lambda_w^2 \Big) \lambda_v^2 = 1, \quad v \in \mathsf{Des}(u). \tag{3.3}$$

To do so, we fix $u \in V$ and set $X_n = \mathsf{Chi}^{\langle 0\rangle}(u) \sqcup \cdots \sqcup \mathsf{Chi}^{\langle n\rangle}(u)$ for $n \geqslant 1$, and $X_0 = \mathsf{Chi}^{\langle 0\rangle}(u)$. Since, by (2.3), $\mathsf{Des}(u) = \bigcup_{n=1}^\infty X_n$, we can construct the required family inductively. For $n = 1$, we put $\lambda_u = \sqrt{\vartheta}$ and choose a family $\{\lambda_v\}_{v\in\mathsf{Chi}(u)} \subseteq (0,\infty)$ such that $\big(\sum_{v\in\mathsf{Chi}(u)} \lambda_v^2\big)\vartheta = 1$ (this is possible because $\mathsf{Chi}(u)$ is nonempty and countable). Fix $n \geqslant 1$, and assume that we already have a family $\{\lambda_v\}_{v\in X_n} \subseteq (0,\infty)$ such that $\lambda_u^2 = \vartheta$ and $\big(\sum_{w\in\mathsf{Chi}(v)} \lambda_w^2\big)\lambda_v^2 = 1$ for all $v \in X_{n-1}$. Then for every $v \in \mathsf{Chi}^{\langle n\rangle}(u)$ we can choose a family $\{\lambda_w\}_{w\in\mathsf{Chi}(v)} \subseteq (0,\infty)$ such that $\big(\sum_{w\in\mathsf{Chi}(v)} \lambda_w^2\big)\lambda_v^2 = 1$. In view of (2.2), this gives us the family $\{\lambda_v\}_{v\in\mathsf{Chi}^{\langle n+1\rangle}(u)}$ such that $\big(\sum_{w\in\mathsf{Chi}(v)} \lambda_w^2\big)\lambda_v^2 = 1$ for all $v \in X_n$. Now by induction we are done.

If $\mathscr{T}$ has a root, then combining (†) and (‡) with (2.5) and Proposition 2.1(i) does the job (the number λ_{root} can be chosen arbitrarily).

Suppose now that $\mathscr{T}$ is rootless. Take $u_1 \in V$ and set $u_2 = \mathsf{par}(u_1)$. By (‡), there exists a family $\{\lambda_v\}_{v\in\mathsf{Des}(u_1)} \subseteq (0,\infty)$ with $\lambda_{u_1} = \frac{1}{\sqrt{2}}$, which satisfies (3.3) with u_1 in place of u. In the next step we construct a new family $\{\lambda_v\}_{v\in\mathsf{Des}(u_2)\setminus\mathsf{Des}(u_1)} \subseteq (0,\infty)$ with $\lambda_{u_2} = \frac{1}{\sqrt{2}}$ such that the extended family

$\{\lambda_v\}_{v\in\mathsf{Des}(u_2)}$ satisfies (3.3) with u_2 in place of u. For this, note that

$$\mathsf{Des}(u_2)\setminus\mathsf{Des}(u_1) \overset{(2.4)}{=} \{u_2\}\sqcup \bigsqcup_{u\in\mathsf{Chi}(u_2)\setminus\{u_1\}} \mathsf{Des}(u). \tag{3.4}$$

Set $\lambda_{u_2}=\frac{1}{\sqrt{2}}$ and choose a family $\{\vartheta_u\}_{u\in\mathsf{Chi}(u_2)\setminus\{u_1\}}\subseteq(0,\infty)$ such that

$$\Big(\sum_{u\in\mathsf{Chi}(u_2)\setminus\{u_1\}}\vartheta_u+\lambda_{u_1}^2\Big)\lambda_{u_2}^2=1. \tag{3.5}$$

Applying (‡) to $u\in\mathsf{Chi}(u_2)\setminus\{u_1\}$ and $\vartheta=\vartheta_u$, we get the family $\{\lambda_v\}_{v\in\mathsf{Des}(u)}\subseteq(0,\infty)$ satisfying (3.2) and (3.3) with $\vartheta=\vartheta_u$. This, together with (3.5), leads to $\big(\sum_{u\in\mathsf{Chi}(u_2)}\lambda_u^2\big)\lambda_{u_2}^2=1$. In view of (3.4), our construction is complete. Applying an induction argument (with $\lambda_{u_n}=\frac{1}{\sqrt{2}}$ for $n\geqslant 2$) and using the fact that $V=\bigcup_{k=0}^\infty\mathsf{Des}(\mathsf{par}^k(u_1))$ (cf. [**8**, Proposition 2.1.6]), we construct a family $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V}\subseteq(0,\infty)$ such that $\zeta_v^2\,\lambda_v^2=1$ for all $v\in V$. This, combined with (†) and Proposition 2.1(i), gives (i).

The "moreover" part follows from (ii) and Proposition 2.1(iii). □

Our method enables us to construct $S_{\boldsymbol{\lambda}}$ with the additional property that $\mathcal{D}(S_{\boldsymbol{\lambda}})\nsubseteq\mathcal{D}(S_{\boldsymbol{\lambda}}^*)$, which is opposite to what happens in Theorem 4.2 below.

Theorem 3.2. *If $\mathscr{T}$ is a directed tree such that* $\mathrm{card}(\mathsf{Chi}(u))=\aleph_0$ *for every $u\in V$, then there exists a family $\boldsymbol{\lambda}=\{\lambda_v\}_{v\in V^\circ}$ of nonzero complex numbers such that $S_{\boldsymbol{\lambda}}$ is injective and densely defined, $\mathcal{D}(S_{\boldsymbol{\lambda}})\nsubseteq\mathcal{D}(S_{\boldsymbol{\lambda}}^*)$ and $\mathcal{D}(S_{\boldsymbol{\lambda}}^2)=\{0\}$.*

Proof. To achieve this, we proceed as in the proof of implication (ii)⇒(i) of Theorem 3.1 with one exception, namely, we strengthen (‡) by requiring, in addition to (3.2) and (3.3), that

$$\sup_{v\in\mathsf{Chi}(u)}\sum_{w\in\mathsf{Chi}(v)}\frac{\lambda_w^4}{1+\lambda_w^2}=\infty. \tag{3.6}$$

This in turn can be deduced from the following fact:

(3.7) for every real number $\alpha>0$, there exists a sequence $\{\lambda_n\}_{n=1}^\infty\subseteq(0,\infty)$ such that $|\lambda_1-\alpha|<1$ and $\sum_{n=1}^\infty\lambda_n^2=\alpha^2$.

Indeed, arguing as in the proof of (‡), we find a family $\{\lambda_v\}_{v\in\mathsf{Chi}(u)}\subseteq(0,\infty)$ such that $\big(\sum_{v\in\mathsf{Chi}(u)}\lambda_v^2\big)\vartheta=1$. Then evidently $\sup_{v\in\mathsf{Chi}(u)}1/\lambda_v^2=\infty$. In the next step, using (3.7), we construct a family $\{\lambda_w\}_{w\in\mathsf{Chi}^{\langle 2\rangle}(u)}$ such that $\big(\sum_{w\in\mathsf{Chi}(v)}\lambda_w^2\big)=1/\lambda_v^2$ for every $v\in\mathsf{Chi}(u)$ and $\sup_{w\in\mathsf{Chi}^{\langle 2\rangle}(u)}\lambda_w^2=\infty$. This, combined with (2.2), implies (3.6). The rest of the proof goes through as for (‡), with hardly any changes. It follows from (2.7) and (3.3) that $\|S_{\boldsymbol{\lambda}}e_w\|^2=1/\lambda_w^2$ for all $w\in\mathsf{Des}(u)$, which together with (3.6) implies that $\sup_{v\in V}\sum_{w\in\mathsf{Chi}(v)}\frac{|\lambda_w|^2}{1+\|S_{\boldsymbol{\lambda}}e_w\|^2}=\infty$. By applying [**8**, Theorem 4.1.1], we deduce that $\mathcal{D}(S_{\boldsymbol{\lambda}})\nsubseteq\mathcal{D}(S_{\boldsymbol{\lambda}}^*)$. Obviously, such $S_{\boldsymbol{\lambda}}$ is never hyponormal. □

4. Hyponormal weighted shifts $S_{\boldsymbol{\lambda}}$ with $\mathcal{D}(S_{\boldsymbol{\lambda}}^2)=\{0\}$

Let us recall a characterization of hyponormality of weighted shifts on directed trees with nonzero weights (in view of [**4**, Proposition 5.3.1], there is no loss of generality in assuming that underlying directed trees are leafless).

Theorem 4.1 ([**8**, Theorem 5.1.2 and Remark 5.1.5])**.** *Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a leafless directed tree $\mathscr{T}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then $S_{\boldsymbol\lambda}$ is hyponormal if and only if*

$$\sum_{v\in\mathsf{Chi}(u)} \frac{|\lambda_v|^2}{\|S_{\boldsymbol\lambda} e_v\|^2} \leqslant 1, \quad u\in V.$$

Now we show that there are hyponormal weighted shifts $S_{\boldsymbol\lambda}$ with $\mathcal{D}(S_{\boldsymbol\lambda}^2) = \{0\}$.

Theorem 4.2. *If $\mathscr{T}$ is a directed tree such that $\operatorname{card}(\mathsf{Chi}(u)) = \aleph_0$ for every $u \in V$, then there exists a family $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ of nonzero complex numbers such that $S_{\boldsymbol\lambda}$ is injective and hyponormal, and $\mathcal{D}(S_{\boldsymbol\lambda}^2) = \{0\}$.*

Proof. We modify the proof of implication (ii)⇒(i) of Theorem 3.1. First we note that for each positive real number r, there exists a sequence $\{r_n\}_{n=1}^\infty \subseteq (0,1)$ such that $\big(\sum_{j=1}^\infty r_j\big) r = 1$ and $\sum_{j=1}^\infty r_j^2 \leqslant 1$ (e.g., $r_j = \frac{1}{rn}$ for $1 \leqslant j \leqslant n-1$, and $r_j = \frac{1}{rn2^{j-n+1}}$ for $j \geqslant n$, where $n \geqslant 2$ is chosen so that $\frac{1}{r^2 n} \leqslant 1$). This fact, when incorporated to the proof of (‡), leads to

(‡‡) for each $(\vartheta, u) \in (0,\infty)\times V$ there exists $\{\lambda_v\}_{v\in\mathsf{Des}(u)} \subseteq (0,1)$ such that $\lambda_u^2 = \vartheta$, $\big(\sum_{w\in\mathsf{Chi}(v)} \lambda_w^2\big)\lambda_v^2 = 1$ and $\sum_{w\in\mathsf{Chi}(v)} \lambda_w^4 \leqslant 1$ for all $v\in\mathsf{Des}(u)$.

If $\mathscr{T}$ has a root, then applying (‡‡) to $u = \mathsf{root}$ and $\vartheta = 1$ we get a family $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ} \subseteq (0,1)$ such that

$$\Big(\sum_{w\in\mathsf{Chi}(v)} \lambda_w^2\Big)\lambda_v^2 = 1 \text{ and } \sum_{w\in\mathsf{Chi}(v)} \lambda_w^4 \leqslant 1 \text{ for all } v\in V. \tag{4.1}$$

Suppose now that $\mathscr{T}$ is rootless. It is easily seen that for every $r\in(0,1)$, there exists a sequence $\{r_j\}_{j=1}^\infty \subseteq (0,1)$ such that $r + \sum_{j=1}^\infty r_j = 2$ and $r^2 + \sum_{j=1}^\infty r_j^2 \leqslant 1$. This fact combined with the proof of Theorem 3.1 (use (‡‡) in place of (‡)) enables us to construct a family $\boldsymbol\lambda = \{\lambda_v\}_{v\in V} \subseteq (0,1)$ that satisfies (4.1).

Since $\operatorname{card}(\mathsf{Chi}(u)) = \aleph_0$ for all $u\in V$, we infer from assertions (i) and (iii) of Proposition 2.1, (4.1) and (†) that $S_{\boldsymbol\lambda}$ is injective and densely defined, and $\mathcal{D}(S_{\boldsymbol\lambda}^2) = \{0\}$. It follows from (2.7) and the equality in (4.1) that $\lambda_v^2 = \|S_{\boldsymbol\lambda} e_v\|^{-2}$ for all $v\in V^\circ$, and thus

$$\sum_{v\in\mathsf{Chi}(u)} \frac{\lambda_v^2}{\|S_{\boldsymbol\lambda} e_v\|^2} = \sum_{v\in\mathsf{Chi}(u)} \lambda_v^4 \overset{(4.1)}{\leqslant} 1, \quad u\in V,$$

which in view of Theorem 4.1 completes the proof. □

Remark 4.3. In view of Theorems 3.2 and 4.2, the weighted shift $S_{\boldsymbol\lambda}$ constructed in the proof of implication (ii)⇒(i) of Theorem 3.1 may satisfy either of the following two conditions: $\mathcal{D}(S_{\boldsymbol\lambda}) \nsubseteq \mathcal{D}(S_{\boldsymbol\lambda}^*)$ or $\mathcal{D}(S_{\boldsymbol\lambda}) \subseteq \mathcal{D}(S_{\boldsymbol\lambda}^*)$. It turns out that this general construction always guarantees that $\mathcal{D}(S_{\boldsymbol\lambda}^*) \nsubseteq \mathcal{D}(S_{\boldsymbol\lambda})$. Indeed, since for a fixed $u\in V$, $\|S_{\boldsymbol\lambda} e_v\|^2 = 1/\lambda_v^2$ for all $v\in\mathsf{Des}(u)$ (cf. (3.3)) and $\sum_{v\in\mathsf{Chi}(u)} \lambda_v^2 < \infty$, we deduce that the function $\phi\colon \mathsf{Chi}(u) \ni v \mapsto \|S_{\boldsymbol\lambda} e_v\| \in \mathbb{C}$ is unbounded, and thus the operator M_u in $\ell^2(\mathsf{Chi}(u))$ of multiplication by ϕ is unbounded (note that the function $\boldsymbol\lambda^u\colon \mathsf{Chi}(u) \ni v \mapsto \lambda_v \in \mathbb{C}$ does not belong to $\mathcal{D}(M_u)$, and so the definition [**8**, (4.2.2)] makes no sense). Applying [**8**, Theorem 4.2.2], we conclude that $\mathcal{D}(S_{\boldsymbol\lambda}^*) \nsubseteq \mathcal{D}(S_{\boldsymbol\lambda})$.

Remark 4.4. It is worth pointing out that if $\mathscr{T}$ is a directed tree such that $\mathrm{card}(\mathsf{Chi}(u)) = \aleph_0$ for every $u \in V$, $S_{\boldsymbol\lambda}$ is a densely defined weighted shifts on $\mathscr{T}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ such that $\mathcal{D}(S_{\boldsymbol\lambda}^2) = \{0\}$ (cf. Theorem 3.1) and $v_0 \in V^\circ$, then the weighted shift $S_{\tilde{\boldsymbol\lambda}}$ on $\mathscr{T}$ with nonzero weights $\tilde{\boldsymbol\lambda} = \{\tilde\lambda_v\}_{v\in V^\circ}$ given by

$$\tilde\lambda_v = \begin{cases} \lambda_v & \text{for } v \neq v_0, \\ \sqrt{1 + \|S_{\boldsymbol\lambda} e_v\|^2} & \text{for } v = v_0, \end{cases}$$

is densely defined, $\mathcal{D}(S_{\boldsymbol\lambda}) = \mathcal{D}(S_{\tilde{\boldsymbol\lambda}})$ (use [**8**, Proposition 3.1.3(i)]), $\mathcal{D}(S_{\boldsymbol\lambda}^*) = \mathcal{D}(S_{\tilde{\boldsymbol\lambda}}^*)$ (use [**8**, Proposition 3.4.1(iv)]), $S_{\tilde{\boldsymbol\lambda}}$ is not hyponormal (use Theorem 4.1) and $\mathcal{D}(S_{\tilde{\boldsymbol\lambda}}^2) = \{0\}$ (use (3.1)). Hence, if $S_{\boldsymbol\lambda}$ is constructed as in the proof of Theorem 4.2, then by Remark 4.3 we have $\mathcal{D}(S_{\tilde{\boldsymbol\lambda}}) \subsetneq \mathcal{D}(S_{\tilde{\boldsymbol\lambda}}^*)$.

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